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1997 J. Phys. A: Math. Gen. 30 L535

(http://iopscience.iop.org/0305-4470/30/15/009)

J. Phys. A: Math. Gen. **30** (1997) L535–L541. Printed in the UK PII: S0305-4470(97)79188-1

LETTER TO THE EDITOR

An orthogonal basis for the B_N-type Calogero model

S Kakei†

Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153, Japan

Received 4 November 1996, in final form 30 May 1997

Abstract. We investigate algebraic structure for the B_N-type Calogero model by using the exchange-operator formalism. We show that the set of Jack polynomials whose arguments are Dunkl-type operators provides an orthogonal basis.

1. Introduction

Among quantum integrable models in one dimension, Calogero–Sutherland-type models have caught renewed interest owing to their relation to fractional statistics. An example of such models is the Calogero model with harmonic potential [1, 2]

$$H_A = \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2\right) + \sum_{j<k}\frac{\beta(\beta-1)}{(x_j-x_k)^2}. \tag{1}$$

The subscript 'A' signifies that this Hamiltonian is invariant under the action of the symmetric group S_N, i.e. the A_{N-1}-type Weyl group. There also exist Calogero-type models associated with other types of Weyl groups [3]. The B_N-invariant counterpart of the Hamiltonian (1) is as follows [4, 5]:

$$H_B = \frac{1}{2}\sum_{j=1}^{N}\left\{-\frac{\partial^2}{\partial x_j^2} + x_j^2 + \frac{\gamma(\gamma-1)}{x_j^2}\right\} + \sum_{j<k}\left\{\frac{\beta(\beta-1)}{(x_j-x_k)^2} + \frac{\beta(\beta-1)}{(x_j+x_k)^2}\right\}. \tag{2}$$

We note that the model associated with the C_N-type Weyl group is equivalent to the B_N-case, and the D_N-type model is obtained by setting $\gamma = 0$. The ground-state wavefunction for this model is [4, 5]

$$\psi_0^{(B)}(x_1, \ldots, x_N) = \prod_{j<k}|x_j^2 - x_k^2|^{\beta}\prod_{j=1}^{N}|x_j|^{\gamma}\prod_{j=1}^{N}\exp(-x_j^2/2). \tag{3}$$

Wavefunctions of the excited states are written as products of $\psi_0^{(B)}$ and some symmetric polynomials. Baker and Forrester [6, 7] obtained an orthogonal basis of such polynomials and named them 'generalized Laguerre polynomials'. It should be noted that the properties of such polynomials have also been studied by van Diejen [8]. In [6], the proof of the orthogonality is based on the orthogonality of another set of polynomials which they called 'generalized Jacobi polynomials'. They obtained the orthogonality of the generalized Laguerre polynomials via some limiting procedure.

† E-mail address: kakei@poisson.ms.u-tokyo.ac.jp

0305-4470/97/150535+07$19.50

Here we make a kind of gauge transformation on the Hamiltonian:

$$\widetilde{H}_B = (\phi_0^{(B)})^{-1} \circ H_B \circ \phi_0^{(B)}$$
$$= \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 - \frac{2\gamma}{x_j}\frac{\partial}{\partial x_j}\right) - \beta\sum_{k\neq j}\frac{1}{x_j^2 - x_k^2}\left(x_j\frac{\partial}{\partial x_j} - x_k\frac{\partial}{\partial x_k}\right) \quad (4)$$

where $\phi_0^{(B)}$ is defined by

$$\phi_0^{(B)}(x_1,\ldots,x_N) = \prod_{j<k}|x_j^2 - x_k^2|^\beta \prod_{j=1}^{N}|x_j|^\gamma. \quad (5)$$

To construct eigenstates of $\widetilde{H}_B$, the exchange-operator formalism is also available [5]. One can construct an analogue of the creation operators $A_j^\dagger$ (for a definition, see (19) later) and show that the wavefunctions of the form $f((A_1^\dagger)^2,\ldots,(A_N^\dagger)^2)\prod_{j=1}^{N}\exp(-x_j^2/2)$ become eigenstates of $\widetilde{H}_B$ if $f(x_1,\ldots,x_N)$ are homogeneous polynomials. However this naive choice of polynomial does not create orthogonal states. In a previous work [9], we have shown that the set of Jack polynomials whose arguments are Dunkl-type operators provides an orthogonal basis for the A_{N-1}-type Calogero model. The aim of this paper is to investigate the B_N-case. We shall show that Jack polynomials also appear in the B_N-case.

2. Dunkl operators and Jack polynomials

In this section, we briefly review the definition of the symmetric and non-symmetric Jack polynomials. In a physical context, Jack polynomials appear as a polynomial part of the wavefunctions for the Sutherland ($1/\sin^2$-interaction) model.

We first introduce the Cherednik operators [10, 11]:

$$\widehat{D}_j^{(A)} = z_j\frac{\partial}{\partial z_j} + \beta\sum_{k(<j)}\frac{z_k}{z_j - z_k}(1 - s_{jk}) + \beta\sum_{k(>j)}\frac{z_j}{z_j - z_k}(1 - s_{jk}) + \beta(j-1) \quad (6)$$

where s_{jk} are elements of the symmetric group S_N (the A_{N-1}-type Weyl group). An element s_{ij} acts on functions of $z_1,\ldots,z_N$ as an operator which permutes arguments z_i and z_j. Since the operators $\widehat{D}_j^{(A)}$ commute with each other, they are diagonalized simultaneously by a suitable choice of bases of $\mathbb{C}[x_1,\ldots,x_N]$ [11, 12]. Such a basis is called the *non-symmetric* Jack polynomial. A non-symmetric Jack polynomial $\mathcal{J}_w^\lambda(x)$, labelled with the partition $\lambda = (\lambda_1,\ldots,\lambda_N)$ and the element $w \in S_N$, is characterized by the following properties [11, 12]:

(i) $\mathcal{J}_w^\lambda(x) = x_w^\lambda + \sum_{(\mu,w')<(\lambda,w)} C_{ww'}^{\lambda\mu} x_{w'}^\mu$,

(ii) $\mathcal{J}_w^\lambda(x)$ is a joint eigenfunction for the operators $\widehat{D}_j^{(A)}$,

where we have used the notation $x_w^\lambda = x_{w(1)}^{\lambda_1}\cdots x_{w(N)}^{\lambda_N}$. To define the ordering $(\mu, w') < (\lambda, w)$, we use the dominance ordering $<_{\mathrm{D}}$ for partitions [13], and the Bruhat ordering $<_{\mathrm{B}}$ for the elements of S_N [14]. We define the ordering as follows:

$$(\mu, w') < (\lambda, w) \Longleftrightarrow \begin{cases} \text{(i)} & \mu <_{\mathrm{D}} \lambda \\ \text{(ii)} & \text{if } \mu = \lambda \text{ then } w' <_{\mathrm{B}} w. \end{cases} \quad (7)$$

We denote the eigenvalues of $\widehat{D}_j^{(A)}$ as $\epsilon_j(\lambda, w)$:

$$\widehat{D}_j^{(A)}\mathcal{J}_w^\lambda(x) = \epsilon_j(\lambda, w)\mathcal{J}_w^\lambda(x). \quad (8)$$

The eigenvalues $\epsilon_j(\lambda, w)$ are all obtained by permutating the components of the multiplet $\{\lambda_{N-j+1} + \beta(j-1)\}_{j=1,\ldots,N}$.

Using the operator $\widehat{D}_j^{(A)}$, we introduce the generating function of symmetric commuting operators [11]:

$$\hat{\Delta}_{\mathrm{S}}^{(A)}(u) = \prod_{j=1}^{N}(u + \widehat{D}_j^{(A)}). \tag{9}$$

Since $\hat{\Delta}_{\mathrm{S}}^{(A)}(u)$ is symmetric in $\widehat{D}_j$, symmetric eigenfunctions are obtained by symmetrizing $\mathcal{J}_w^{\lambda}(x)$, which are nothing but the Jack symmetric polynomials $J_\lambda(x)$. Eigenvalues of $\hat{\Delta}_{\mathrm{S}}^{(A)}(u)$ are then given by

$$\hat{\Delta}_{\mathrm{S}}^{(A)}(u) J_\lambda(x) = \prod_{j=1}^{N}\{u + \lambda_{N-j+1} + \beta(j-1)\} J_\lambda(x). \tag{10}$$

We note that all the eigenvalues of $\hat{\Delta}_{\mathrm{S}}^{(A)}(u)$ are distinct for generic values of u.

We then introduce the B_N-type Dunkl operators [5, 15]

$$D_j^{(B)} = \frac{\partial}{\partial x_j} + \beta \sum_{k(\neq j)} \left\{ \frac{1 - s_{jk}}{x_j - x_k} + \frac{1 - t_j t_k s_{jk}}{x_j + x_k} \right\} + \gamma \frac{1 - t_j}{x_j} \tag{11}$$

where s_{jk} and t_j are elements of the B_N-type Weyl group. An element s_{ij} acts the same as in the A_{N-1}-case and t_j acts as a sign change, i.e. it replaces the coordinate x_j by $-x_j$. Commutation relations of the B_N-type Dunkl operators are

$$[D_i^{(B)}, D_j^{(B)}] = 0 \qquad s_{ij} D_j^{(B)} = D_i^{(B)} s_{ij} \qquad s_{ij} D_k^{(B)} = D_k^{(B)} s_{ij} \qquad (k \neq i, j)$$
$$t_j D_j^{(B)} = -D_j^{(B)} t_j \qquad t_j D_k^{(B)} = D_k^{(B)} t_j \qquad (k \neq j)$$
$$[D_i^{(B)}, x_j] = \delta_{ij}\left(1 + \beta \sum_{k(\neq j)} (s_{jk} + t_j t_k s_{jk}) + 2\gamma t_j\right) - (1 - \delta_{ij})\beta(s_{ij} - t_i t_j s_{ij}). \tag{12}$$

We denote the algebra generated by the elements x_j, $D_j^{(B)}$, s_{ij} and t_j as $\mathcal{A}_{\mathrm{S}}^{(B)}$. We introduce an $\mathcal{A}_{\mathrm{S}}^{(B)}$-module $\mathcal{F}_{\mathrm{S}}^{(B)}$ ('Fock space' for $\mathcal{A}_{\mathrm{S}}^{(B)}$) generated by the vacuum vector $|0\rangle_{\mathrm{S}} = 1$:

$$\mathcal{F}_{\mathrm{S}}^{(B)} = \mathbb{C}[x_1^2, \ldots, x_N^2]|0\rangle_{\mathrm{S}}. \tag{13}$$

The elements $D_j^{(B)}$ of $\mathcal{A}_{\mathrm{S}}^{(B)}$ annihilate the vacuum vector, and s_{ij}, t_j preserve $|0\rangle_{\mathrm{S}}$:

$$D_j|0\rangle_{\mathrm{S}} = 0 \qquad s_{ij}|0\rangle_{\mathrm{S}} = |0\rangle_{\mathrm{S}} \qquad t_j|0\rangle_{\mathrm{S}} = |0\rangle_{\mathrm{S}}. \tag{14}$$

We then define Cherednik-type commuting operators associated with (11):

$$\begin{aligned}\widehat{D}_j^{(B)} &= x_j D_j^{(B)} + \beta \sum_{k(<j)} (s_{jk} + t_j t_k s_{jk}) \\ &= x_j \frac{\partial}{\partial x_j} + \beta \sum_{k(<j)} \left\{ \frac{x_k}{x_j - x_k}(1 - s_{jk}) - \frac{x_k}{x_j + x_k}(1 - t_j t_k s_{jk}) \right\} \\ &\quad + \beta \sum_{k(>j)} \left\{ \frac{x_j}{x_j - x_k}(1 - s_{jk}) + \frac{x_j}{x_j + x_k}(1 - t_j t_k s_{jk}) \right\} \\ &\quad + 2\beta(j-1) + \gamma(1 - t_j).\end{aligned} \tag{15}$$

We introduce the notation $\mathrm{Res}^{(t)}(X)$ which means the action of the operator X is restricted to the functions with symmetry $t_j f(x) = f(x)$. Under this restriction, the action of the operator $\widehat{D}_j^{(B)}$ is reduced to the following form:

$$\mathrm{Res}^{(t)}(\widehat{D}_j^{(B)}) = x_j \frac{\partial}{\partial x_j} + 2\beta \sum_{k(<j)} \frac{x_k^2}{x_j^2 - x_k^2}(1 - s_{jk})$$

$$+2\beta \sum_{k(>j)} \frac{x_j^2}{x_j^2 - x_k^2}(1 - s_{jk}) + 2\beta(j-1). \tag{16}$$

Comparing (16) with (6), we find that $\mathrm{Res}^{(t)}(\widehat{D}_j^{(B)})$ is equivalent to $2\widehat{D}_j^{(A)}$ if we make a change of variables $z_j = x_j^2/2$. If we define the operator $\hat{\Delta}_{\mathrm{S}}^{(B)}(u)$ as

$$\hat{\Delta}_{\mathrm{S}}^{(B)}(u) = \prod_{j=1}^{N}(u + \widehat{D}_j^{(B)}) \tag{17}$$

we then have the following equation by using the correspondence between $\mathrm{Res}^{(t)}(\widehat{D}_j^{(B)})$ and $2\widehat{D}_j^{(A)}$:

$$\hat{\Delta}_{\mathrm{S}}^{(B)}(u) J_\lambda\left(x_1^2/2, \ldots, x_N^2/2\right) = \prod_{j=1}^{N}\left\{u + 2\lambda_{N-j+1} + 2\beta(j-1)\right\} J_\lambda\left(x_1^2/2, \ldots, x_N^2/2\right). \tag{18}$$

3. B_N-type Calogero model

We now turn to the B_N-type Calogero model. We introduce an analogue of creation and annihilation operators [5]:

$$A_j^\dagger = \frac{1}{\sqrt{2}}(-D_j^{(B)} + x_j) \qquad A_j = \frac{1}{\sqrt{2}}(D_j^{(B)} + x_j). \tag{19}$$

By a direct calculation, we can show that the operator $A_j^\dagger$ is adjoint of A_j with respect to the scalar product

$$(f, g)_B = \int_{-\infty}^{\infty} f(x_1, \ldots, x_N) g(x_1, \ldots, x_N)(\phi_0^{(B)})^2 \prod_{j=1}^{N} \mathrm{d}x_j. \tag{20}$$

We call an algebra generated by A_j, $A_j^\dagger$, s_{ij} and t_j as $\mathcal{A}_{\mathrm{C}}^{(B)}$. Since the commutation relations of these operators are the same as those of x_j and $D_j^{(B)}$, we can define an isomorphism of $\mathcal{A}_{\mathrm{S}}^{(B)}$ to $\mathcal{A}_{\mathrm{C}}^{(B)}$ as follows:

$$\sigma(x_j) = A_j^\dagger \qquad \sigma(D_j^{(B)}) = A_j. \tag{21}$$

The Fock space for $\mathcal{A}_{\mathrm{C}}^{(B)}$ is constructed in the same way as for $\mathcal{F}_{\mathrm{S}}^{(B)}$:

$$\mathcal{F}_{\mathrm{C}}^{(B)} = \mathbb{C}[(A_1^\dagger)^2, \ldots, (A_N^\dagger)^2]|0\rangle_{\mathrm{C}} \tag{22}$$

with $|0\rangle_{\mathrm{C}} = \prod_{j=1}^{N} \exp(-x_j^2/2)$. The elements A_j of $\mathcal{A}_{\mathrm{C}}^{(B)}$ annihilate the vacuum vector, and s_{ij}, t_j preserve $|0\rangle_{\mathrm{C}}$:

$$A_j|0\rangle_{\mathrm{C}} = 0 \qquad s_{ij}|0\rangle_{\mathrm{C}} = |0\rangle_{\mathrm{C}} \qquad t_j|0\rangle_{\mathrm{C}} = |0\rangle_{\mathrm{C}}. \tag{23}$$

Comparing (23) with (14), we know that the isomorphism σ can be extended to the isomorphism of the Fock spaces:

$$\sigma(|0\rangle_{\mathrm{S}}) = |0\rangle_{\mathrm{C}} \qquad \sigma(a|v\rangle) = \sigma(a)\sigma(|v\rangle) \tag{24}$$

for $a \in \mathcal{A}_{\mathrm{S}}^{(B)}$ and $|v\rangle \in \mathcal{F}_{\mathrm{S}}^{(B)}$.

Applying this isomorphism to (18), we obtain the following equation:

$$\hat{\Delta}_{\mathrm{C}}^{(B)}(u) J_\lambda((A_1^\dagger)^2/2, \ldots, (A_N^\dagger)^2/2)|0\rangle_{\mathrm{C}}$$

$$= \prod_{j=1}^{N} \left\{u + 2\lambda_{N-j+1} + 2\beta(j-1)\right\} J_\lambda \left((A_1^\dagger)^2/2, \ldots, (A_N^\dagger)^2/2\right) |0\rangle_{\mathrm{C}} \tag{25}$$

where we define $\hat{\Delta}_{\mathrm{C}}^{(B)}(u)$ as

$$\hat{\Delta}_{\mathrm{C}}^{(B)}(u) = \sigma(\hat{\Delta}_{\mathrm{S}}^{(B)}(u)) = \prod_{j=1}^{N}(u + \hat{h}_j^{(B)}) \tag{26}$$

with

$$\hat{h}_j^{(B)} = \sigma(\widehat{D}_j^{(B)}) = A_j^\dagger A_j + \beta \sum_{k(<j)} (s_{jk} + t_j t_k s_{jk}). \tag{27}$$

We note that the operator $\hat{h}_j^{(B)}$ is self-adjoint with respect to (20). The transformed Hamiltonian $\widetilde{H}_B$ is related to (26) as follows; if we denote the $(N-j)$th coefficient of $\hat{\Delta}_{\mathrm{C}}^{(B)}(u)$ as $I_{\mathrm{C},j}^{(B)}$, then $\widetilde{H}_B$ is obtained from $I_{\mathrm{C},1}^{(B)}$ after restricting to the B_N-invariant subspace:

$$\mathrm{Res}(I_{\mathrm{C},1}^{(B)}) = \mathrm{Res}\left(\sum_{j=1}^{N} \hat{h}_j^{(B)}\right) = \widetilde{H}_B - \frac{N}{2} - \gamma N. \tag{28}$$

From (25) we find that all the eigenvalues of $\hat{\Delta}_{\mathrm{C}}^{(B)}(u)$ are distinct. On the other hand, the operator $\hat{\Delta}_{\mathrm{C}}^{(B)}(u)$ is self-adjoint with respect to the scalar product (20). From these facts, we conclude that the vectors

$$|\lambda\rangle = J_\lambda((A_1^\dagger)^2/2, \ldots, (A_N^\dagger)^2/2)|0\rangle_{\mathrm{C}} \tag{29}$$

form an orthogonal basis with respect to the scalar product (20). We note that the polynomial parts of the basis (29) are equivalent to the generalized Laguerre polynomials introduced by Baker and Forrester up to a constant.

Baker and Forrester [7] also introduced 'non-symmetric generalized Laguerre polynomials'. In our formulation, such polynomials are related to joint eigenfunctions of the operators $\hat{h}_j^{(B)}$, which are of the form $\mathcal{J}_w^\lambda((A_1^\dagger)^2/2, \ldots, (A_N^\dagger)^2/2)|0\rangle_{\mathrm{C}}$. The non-symmetric generalized Laguerre polynomials are polynomial parts of these eigenfunctions.

In conclusion, we have constructed an operator expression of the orthogonal basis for the B_N-type Calogero model by using Jack polynomials whose arguments are the Dunkl-type creation and annihilation operators. We stress that our proof of orthogonality is algebraic and does not make use of the limiting procedure.

Appendix

In this appendix we investigate the one-variable case in more detail to clarify the relationship to the Laguerre polynomials.

For $N = 1$ case, Hamiltonian (2) is reduced to

$$\widehat{H} = \frac{1}{2}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{\gamma(\gamma-1)}{x^2}\right\}. \tag{A1}$$

The ground-state wavefunction is $\psi_0(x) = |x|^\gamma \exp(-x^2/2)$, whose eigenvalue is $1/2 + \gamma$ (we omit the normalization constant).

On the other hand, the creation and annihilation operators (19) are reduced to

$$A^\dagger = \frac{1}{\sqrt{2}}\left\{-\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{\gamma}{x}(1-\hat{t})\right\} \qquad A = \frac{1}{\sqrt{2}}\left\{\frac{\mathrm{d}}{\mathrm{d}x} + x + \frac{\gamma}{x}(1-\hat{t})\right\} \tag{A2}$$

where $\hat{t}$ is the reflection operator $\hat{t}f(x) = f(-x)$. The Hamiltonian (A1) and the operators (A2) are related as follows:

$$\widehat{H} = |x|^{\gamma} \circ \tfrac{1}{2}\,\mathrm{Res}(A^{\dagger}A + AA^{\dagger}) \circ |x|^{-\gamma}$$

where Res X means that action of the operator X is restricted to even functions.

Wavefunctions for excited states can be constructed by using a gauge-transformed version of (A2), i.e.

$$\hat{A}^{\dagger} = |x|^{\gamma} \circ A^{\dagger} \circ |x|^{-\gamma} = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + x + \frac{\gamma}{x}\,\hat{t}\right)$$

$$\hat{A} = |x|^{\gamma} \circ A \circ |x|^{-\gamma} = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{\gamma}{x}\,\hat{t}\right)$$

which obey the commutation relations

$$[\widehat{H}, \hat{A}^{\dagger}] = \hat{A}^{\dagger} \qquad [\widehat{H}, \hat{A}] = -\hat{A}. \tag{A3}$$

It should be noted that these creation and annihilation operators have been introduced by Yang [16].

To preserve the symmetry $\psi(x) = \psi(-x)$, we apply $\hat{A}^{\dagger}$ an even number of times to the ground-state wavefunction $\psi_0(x)$:

$$\psi_{2n}(x) = (\hat{A}^{\dagger})^{2n}\psi_0(x).$$

This formula is the $N = 1$ counterpart of (29). From (A3), it follows that $\psi_{2n}(x)$ are also eigenfunctions of $\widehat{H}$:

$$\widehat{H}\psi_{2n}(x) = (2n + 1/2)\psi_{2n}(x). \tag{A4}$$

The wavefunctions $\psi_{2n}(x)$ are expressed as a product of some polynomials $f_{2n}(x)$ and the ground-state wavefunction $\psi_0(x)$. Rewriting (A4), one can obtain the following differential equation for $f_{2n}(x)$:

$$\frac{\mathrm{d}^2 f_{2n}}{\mathrm{d}x^2} - \left(2x - \frac{2\gamma}{x}\right)\frac{\mathrm{d}f_{2n}}{\mathrm{d}x} + 4nf_{2n} = 0.$$

Making a change of variable $y = x^2$, we obtain

$$y\frac{\mathrm{d}^2 f_{2n}}{\mathrm{d}y^2} + \left(\frac{1}{2} + \gamma - y\right)\frac{\mathrm{d}f_{2n}}{\mathrm{d}y} + nf_{2n} = 0$$

which coincides with the differential equation for the Laguerre polynomials. Hence we conclude that $f_{2n}(x)$ can be written by using the Laguerre polynomials:

$$f_{2n}(x) = n!(-2)^n L_n^{(\gamma-1/2)}(x^2).$$

Since $\psi_{2n}(x)$ are even functions, the operators $(\hat{A}^{\dagger})^2$ and $\hat{A}^2$ act equivalently to

$$B^{+} = \mathrm{Res}((\hat{A}^{\dagger})^2) = \frac{1}{2}\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + x^2 - 1 - \frac{\gamma(\gamma-1)}{x^2}\right\}$$

$$B^{-} = \mathrm{Res}(\hat{A}^2) = \frac{1}{2}\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}}{\mathrm{d}x} + x^2 + 1 - \frac{\gamma(\gamma-1)}{x^2}\right\}$$

respectively. We remark that the operators B^{+} and B^{-} have been introduced by Perelomov [17].

The operators B^{+} and B^{-} give the recursion relations for the wavefunctions

$$B^{+}\psi_{2n} = \psi_{2n+2} \qquad B^{-}\psi_{2n} = 4n(n - \tfrac{1}{2} + \gamma)\psi_{2n-2} \tag{A5}$$

where the constant factor of the second relation is determined by comparing the coefficient of $x^{2n-2}\psi_0(x)$. One can obtain recursion relations for the Laguerre polynomials by rewriting (A5) [17]:

$$\left\{y\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\left(\frac{1}{2}+\gamma-2y\right)\frac{\mathrm{d}}{\mathrm{d}y}+2y-\frac{1}{2}-\gamma\right\}L_n^{(\gamma-1/2)}(y)=-(n+1)L_{n+1}^{(\gamma-1/2)}(y)$$
$$\left\{y\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\left(\frac{1}{2}+\gamma\right)\frac{\mathrm{d}}{\mathrm{d}y}\right\}L_n^{(\gamma-1/2)}(y)=-\left(n-\frac{1}{2}+\gamma\right)L_{n-1}^{(\gamma-1/2)}(y).$$

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